Chapter Four: Grade 11 Learners' Adaptive Reasoning Proficiencies in Solving Euclidean Geometry Problems

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Introduction

Until recently, successful mathematics learning meant mastery of arithmetic computational skills (Findell et al. 2001; Moodley 2008). However, there has been a discussion about what it means to be successful in mathematics education (Moodley 2008). It has been argued that learner success in learning mathematics strongly depends on their proficiency in terms of conceptual and procedural competencies, expertise in reasoning and problem solving and facility in the subject, for example (Ally 2011; Findell et al. 2001; Ho 2020). Mathematical proficiency, a central aspect in mathematical understanding, reasoning and problem solving, is a requirement for learner competitiveness in today's and tomorrow's world (Moodley 2008; Syukriani et al. 2017).

Mathematical proficiency, which is synonymous with problem-solving proficiency, has been thought of as consisting of five mutually interdependent strands (Findell et al. 2001). The strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. During mathematical problem solving, adaptive reasoning interacts with other mathematical proficiencies, especially when a solution strategy requires the use of procedures for calculation, measurement or display. In such a situation, adaptive reasoning is essential to identify an appropriate procedure (Syukriani et al. 2017). In favour of depth, the focus in the presented work is on adaptive reasoning.

Adaptive reasoning includes knowledge of how to justify the steps taken in solving a problem logically, justify conclusions and assess the solution steps (Ansari et al. 2020; Mulyayunita and Nurjanah 2019). Learners with adaptive reasoning can also explore different solution strategies, comprehend the logical structure of a proposed proof and identify logical inconsistencies (Siegfried 2012). Adaptive reasoning plays a critical role in increasing learners' deductive reasoning and their capacity to think, particularly in geometry, but also in other branches of mathematics (Mudaly and De Villiers 2004; Mulyayunita and Nurjanah 2019).

The question can be asked as to how proficient the school output is in South Africa, for example, particularly in adaptive reasoning, and in mathematics in general. In this regard, it is helpful to first consider the Curriculum and Assessment Policy Statement (CAPS) for Mathematics Grades 10–12 in South Africa. Amongst the specific aims of CAPS for mathematics, teaching is the provision of opportunities for learners to develop the ability to be methodical, generalise, and generate, prove or justify conjectures (Department of Basic Education [DBE] 2011). Also included is assisting learners to recognise and resolve issues and make judgements using both analytical and imaginative thinking. There is, however, evidence of an undesirable state of learner performance in Euclidean geometry (EG), in examination results and research studies.

Learners find it difficult to solve mathematical problems, particularly those involving EG (DBE 2020, 2021, 2022; Dongwi 2014; Malatjie and Machaba 2019; Mthethwa 2015; Naidoo and Kapofu 2020). Grade 12 examination (matric) results show that most learners are not mathematically proficient by the end of the compulsory phase of schooling (DBE 2020; Dhlamini and Luneta 2016; Moodley 2008). In this regard, Table 4.1 contains information about the performance of Grade 12 learners in EG in matric examinations compared to other aspects of geometry and mathematics.

Table 4.1: Grade 12 learners' performance in EG and three other areas of mathematics

Topic	2020	2021	2022
Analytical geometry	52 %	50%	57 %
Euclidean geometry	41.3 %	38%	34.7 %
Statistics	75.5 %	75 %	60.4 %
Trigonometry	46.7%	38.5%	36.7 %
Source: DBE diagnostic reports 2020–2022			

Table 4.1 specifically presents the results of Grade 12 learners in matric exams for Mathematics Paper 2 for three consecutive years and in four curriculum topics, including EG. The proportion of learners passing in EG is relatively low.

Many learners perceive EG as the most challenging part of the mathematics curriculum (Patkin and Lavenberg 2012). In a study by Ali et al. (2014), 92 of the 120 learners failed an EG test. It has also been noted that many high school learners struggle to construct and characterise features of plane geometry effectively and construct adequate proofs in geometry (Dongwi 2014). In the Vhembe East District, South Africa, the first author of the current chapter (Roger Mayani) noted in his experience as a mathematics teacher that Grade 11 learners frequently skip or score poorly in the EG sections during the year and in the final exam.

There have been some studies on EG and the difficulties in learning EG in South Africa. The studies have focused on learner deduction levels (Masilo 2018), EG terminology (Alex and Mammen 2018) and understanding of basic EG concepts (Ngirishi and Bansilal 2019). Researchers have paid more attention to conceptual understanding in EG, at the expense of adaptive reasoning proficiencies. The presented study contributes to bridging this gap in research. The question that guided the current study was, 'How proficient are Grade 11 learners in the Vhembe East District of South Africa in adaptive reasoning and EG problem solving?' The purpose of the study was to yield a description of the mathematical proficiency of the learners.

Conceptual framework

The purpose of the presented study relates to adaptive reasoning proficiency with a focus on Grade 11 EG learning in South Africa. Thus, it is helpful to outline Grade 11 EG in the context of the Grades R–12 EG mathematics curriculum of South Africa. Also helpful is a more detailed discussion of adaptive reasoning in mathematics.

EG in the South African school curriculum

In Grade R, learners learn to describe the relative position of objects and follow directions to move around the classroom (DBE 2012). In Grades 1–3, learners consider different views of the same object and follow directions on an informal map. In Grades 4–6, learners, amongst other aspects, sort and compare 2-D shapes and draw the shapes on grid paper. In addition, they use a pair of compasses to draw circles, patterns in or with circles. In Grades 8–9, learners distinguish different types of triangles and describe the properties of congruent and similar shapes. They also solve problems involving triangles and quadrilaterals, classify 3D objects, build 3D models and perform transformations on a co-ordinate plane. In Grade 10, learners investigate line segments, define special quadrilaterals, investigate and make conjectures about the properties of the quadrilaterals and prove the conjectures. Additionally, the learners solve problems and prove riders. Also involved are theorems and proofs.

In Grade 11, the focus of the presented study is that learners investigate and prove theorems of the geometry of circles and their converses and use them to solve problems. The theorems are:

- The line drawn from the centre of a circle perpendicular to a chord bisects the chord.
- The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- The perpendicular bisector of a chord passes through the centre of the circle.
- The angle subtended by an arc at the centre of a circle is double

the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).

- Angles subtended by a chord of the circle, on the same side of the chord, are equal.
- The opposite angles of a cyclic quadrilateral are supplementary.
- An exterior angle of a cyclic quadrilateral is equal to an angle in the alternate segment.
- Two tangents drawn to a circle from the same point outside the circle are equal in length.
- The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment, (DBE 2011: 34)

In Grade 12, learners prove that a line parallel to one side of a triangle divides the other two sides proportionally, equiangular triangles are similar, triangles with sides in proportion are similar, and the Pythagorean Theorem (in a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides) using similar triangles. Furthermore, the learners use these theorems and their converses to solve and prove riders (DBE 2011). Outlined above is the EG content framework for the development of adaptive reasoning proficiency, in the Vhembe East District, like elsewhere in South Africa.

Adaptive reasoning in mathematics

It has been noted that adaptive reasoning involves logical thinking about the relationship between concepts and situations (Findell et al. 2001). Also included is the careful consideration of alternative solution pathways. Adaptive reasoning also includes the ability to justify problem-solving steps, knowing when the steps are incorrect and how to justify conclusions (Ansari et al. 2020; Mulyayunita and Nurjanah 2019). Hence, it encompasses not only deductive reasoning, but also inductive and intuitive reasoning using patterns, analogies and metaphors (Susilawati et al. 2021). To demonstrate adaptive reasoning, learners must think logically about problems, estimate and reflect on them and provide justifications for their problem-solving.

Learners show adaptive reasoning proficiency in EG by investigating, proving and utilising a range of geometry of the circle theorems and their converses, to solve problems. In this context, learners must justify their statements, prove the theorems, understand concepts, apply them to novel problem situations and utilise inductive and deductive reasoning. While adaptive reasoning consists of inductive and deductive reasoning (Ansari et al. 2020; Findell et al. 2001), inductive reasoning involves identifying and applying patterns in mathematical problem solving (Fischbein, as cited in Ansari et al. 2020). On the other hand, deductive reasoning is the learner's ability to make predictions, present reasoning and examine an argument.

Five categories of adaptive reasoning proficiency have been identified, and consist of proposing predictions or conjectures, providing reasons for given solutions, finding patterns in a problem, examining the validity of an argument and drawing conclusions from a statement (Ansari et al. 2020; Findell et al. 2001). Table 4.2 outlines categories and levels of adaptive reasoning proficiency.

Table 4.2: Adaptive reasoning categories and proficiency levels

Category	Level of Proficiency		
	Excellent (E)	**Moderate (M)**	**Poor (P)**
1. Ability to propose a conjecture	Correct and complete	Less complete	Presenting wrong conjecture
2. Ability to find the pattern of a problem	Correct with calculation	Less complete with some miscalculations	Wrong pattern and calculation
3. Ability to present reasoning for the solution	Correct and complete	Managed to present the reasoning	Incorrect or providing incorrect reasoning
4. Ability to draw correct conclusions	Correct and complete	Managed to write the conclusion	Without conclusion
5. Ability to examine the validity of an argument	Correct with calculation	Less correct (some miscalculations)	Miscalculation

Source: Adapted from Ansari et al. (2020)

The Rubric of Mathematical Adaptive Reasoning (RMAR) (Table 4.2) has been used to investigate the adaptive reasoning proficiency of Grade 8 learners in a junior high school in Indonesia when answering a test of adaptive reasoning proficiency in algebra, plane and 3-D geometry questions (Ansari et al. 2020).

Research methodology

Within the framework of a qualitative approach, the current study used a descriptive case study design. A descriptive case study is research that is an in-depth examination of a few units of study over a period or across multiple periods of time (Creswell 2014; Creswell and Creswell 2018; Leavy 2017).

Prior to the data collection, approval was obtained from the ethics committee of the University of Pretoria and the Limpopo Department of Basic Education. The data collection took place in ten public schools in the Vhembe East District, Limpopo province, South Africa. The schools that participated in the research study were named School A to School J. Twenty learners were sampled from each school to participate in the research study, making a total of 200 learners.

Under the supervision of the first author (Roger Mayani), a Euclidean Geometry Proficiency Test (EGPT) was used to acquire qualitative data about mathematical problem solving. The EGPT was aligned with the RMAR (Table 4.2) to cover all the categories of proficiencies. The tasks on the EGPT classified as predominantly adaptive reasoning tasks were numbered: 1.2; 2.1; 2.2; 3.2; 4.1.1; 4.1.2; 5.1 – 5.4; 6.1 and 6.2.

This study used the RMAR framework to analyse the learners' solutions to the problems. This ensured that the data was interpreted in accordance with the research question. The analysis was done using descriptive statistics. Specifically, the current study employed tables to find the frequencies of learners' problem-solving proficiency levels, to illustrate the analysis of learners' levels of mathematical proficiencies and to calculate the mean scores obtained by learners in adaptive reasoning. In the content analysis, excerpts of the learners' responses to the questions were selected for use as concrete examples in the presentation of the findings.

Findings

The findings, based on three of the problems addressed by the learners, are exemplified below. The questions are 1.2, 2.1 and 5.1. Question 1.2 is shown in Figure 4.1.

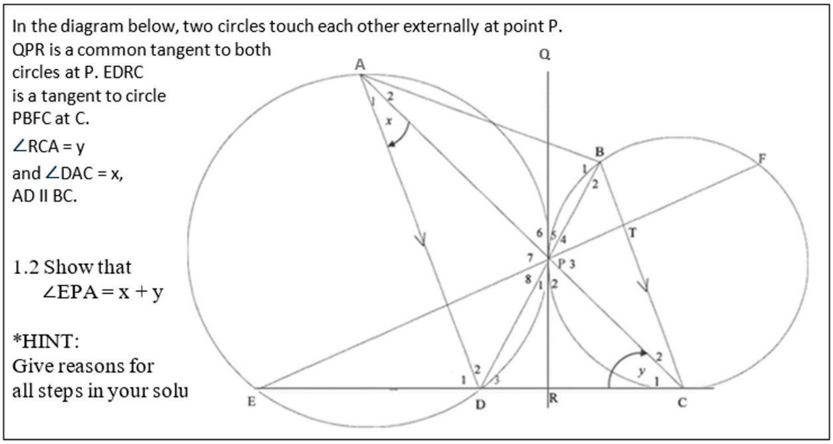


Figure 4.1: Question 1.2

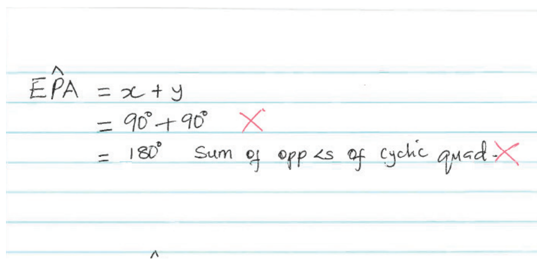
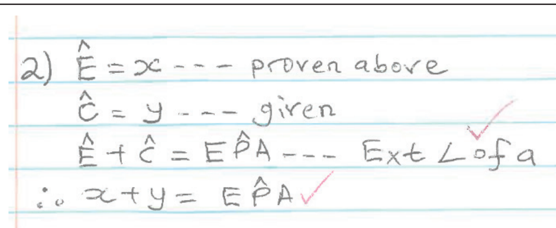
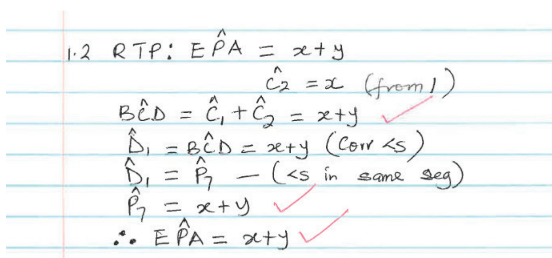
In Question 1.2, learners were asked to show that ∠EPA = x + y. The solution to this question should follow the following reasoning pattern:

(1) Make connection with the previous question by ∠C2 = x (from 1.1).
(2) Establish the correct pattern ∠D1 = ∠DCB = x + y with the reason that they are corresponding angles, AD II BC.
(3) State that ∠D1 = P7 with the reason that they are angles in the same segment.
(4) Use deductive reasoning to conclude that ∠P7 = ∠EPA = x + y.

In Question 1.2, 166 learners, making up to 83 per cent of the participants, were found to be poor (Table 4.2). The learners' solutions showed that the learners lacked adaptive reasoning proficiencies. The solutions did not show any conjecture in them; there were no patterns of the problem; the learners failed to examine the validity of their argument; the conclusions were written without any justifications and did not show a logical relationship between concepts and situations. Ten learners, accounting for 5 per cent of the participants, were ranked moderate, as their ability to propose a conjecture were not complete; the patterns of the problem were not correct; the learners managed to write the conclusions, however, the solutions were found to be less valid because the patterns followed had mistakes. Twenty-four learners, making up to 12 per cent of the participants were classified excellent: the learners wrote correct patterns. Moreover, the learners proposed correct statements and their conclusions were made from correct, logical and complete reasoning. This proves that the learners could examine the validity of their arguments.

Table 4.3 shows examples of learners' work at different levels of adaptive reasoning in Question 1.2

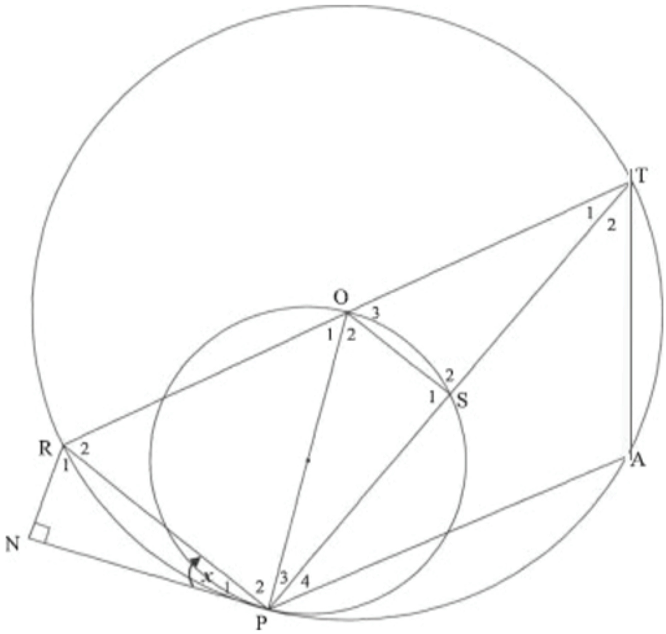
Table 4.3: Learner proficiency level and excerpts of solutions to Question 1.2

Learner	**Level of Proficiency**	**Excerpt**
A	**Poor**	
B	**Moderate**	
C	Excellent	

The solution of Learner A showed that the learner lacked several adaptive reasoning proficiencies. The learner started with the conclusion that ∠EPA = x + y = 90°+90°= 180° with the wrong reason of sum of angles of cyclic quadrilateral; the learner's solution does not show any conjecture in it; there is no pattern of the problem; the learner failed to examine the validity of their argument and the conclusion was written without proposing a conjecture and without finding the pattern of the problem. For these reasons, Learner A was ranked poor in adaptive reasoning. In the solution of Learner B, the pattern of the problem is partly correct: the learner proposed that ∠E = x then in the second line of their answer, they wrote ∠C = y instead of ∠C1 = y; the statement ∠EPA = ∠E + ∠C is partly correct because ∠EPA = ∠E + ∠C1; in their attempted solution, the learner failed to establish that ∠EPA = ∠P7 = x + y. The learner's conclusion is, therefore, less valid because the pattern followed contained mistakes. For these reasons, Learner B was found to be moderate in adaptive reasoning. The solution of Learner C showcases as excellent in adaptive reasoning proficiencies. The learner wrote a correct pattern: ∠C2 = x; ∠BDC = ∠C1 + ∠C2 = x + y; ∠D1 = ∠BCD = x + y; ∠D1 = P7 = x + y; ∠EPA = x + y. Moreover, the learner proposed a correct statement: ∠BCD = ∠C1 + ∠C2 = x + y; their conclusion was made from a correct and complete reasoning. This proves that the learner examined the validity of their argument.

In Question 2.1, learners were asked to prove that PR bisected ORN. The solution to this problem should follow the following reasoning pattern: (1) find the correct pattern to the problem and solve for ∠R1: ∠R1 + ∠N + ∠P1 = 180° (sum of ∠s in a Δ); ∠R1 + 90°+ x = 180°; ∠R1 = 180° - 90°- x; ∠R1 = 90°- x; (3) present the correct statement and correct reason that ∠NPO = 90° (tan ⊥ radius); (4) present the second pattern and solve for ∠P2: ∠NPO = ∠P1 + ∠P2; ∠P2 = 90°- x; (5) establish that ∠R2 = ∠P2 (∠s opp. = sides); (6) deduct that ∠R2 = 90°- x; (7) using deductive reasoning to conclude that ∠R1 = ∠R2 = 90°- x; PR bisects ORN (∠R1 = ∠R2).

O is the centre of the circle RTAP. OP is the diameter of the smaller circle PSO.
NP is a tangent to both circles at P. RN ⊥ NP. Let $P_1 = x$.



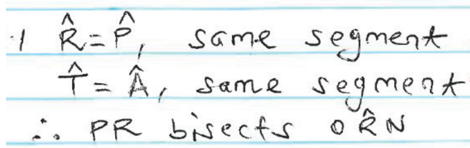
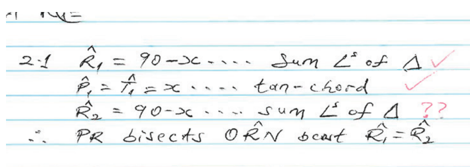
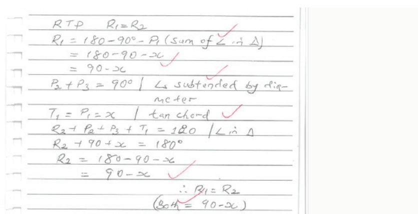
2.1. Prove that PR bisects ∠ORN.

Figure 4.2: Question 2.1

In Question 2.1, 158 learners (79 per cent) were ranked poor in adaptive reasoning. The learners presented wrong conjectures with incorrect reasons, and their patterns of the problems were incorrect. Moreover, the conclusions were written following incorrect reasoning. Furthermore, the learners failed to examine the validity of their arguments. Twelve learners (6 per cent) were moderate in adaptive reasoning: the learners proposed less complete conjectures, and the reasoning pattern to support conclusions was unclear. Moreover, the learners managed to examine the validity of their arguments, but not completely. Thirty learners (15 per cent) were excellent in adaptive reasoning. These learners were able to propose correct statements, and they were able to write the correct patterns with correct calculations. Their conclusions were drawn from a correct and complete reasoning pattern. Moreover, they were able to examine the validity of their arguments.

Table 4.4 shows examples of learners' work at different levels of adaptive reasoning on Question 2.1

Table 4.4: Learner proficiency level and excerpts of solutions to Question 2.1

Learner	Level of Proficiency	Excerpt
D	Poor	
E	Moderate	
F	Excellent	

Learner D proposed a wrong conjecture ∠R = ∠P, with incorrect reasoning (same segment); the learner's pattern of the problem is incorrect: ∠R = ∠P, ∠T = ∠A. Moreover, the conclusion was written from incorrect reasoning. For these reasons, learner D was declared poor in adaptive reasoning proficiency. Learner E proposed a correct statement ∠R1 = 90°- x and ∠R2 = 90°- x, however, the reasoning pattern to reach the statement is not clear in their attempted solution. Moreover, the learner managed to present the reasoning for their solution, but not completely. For these reasons, Learner E was ranked moderate in adaptive reasoning. Learner F was able to propose the correct statement that ∠T1 = P1 = x; the learner wrote the correct pattern $\angle R2 + \angle T1 + \angle RPT = 180°$ (sum of $\angle$s of triangle) with correct calculations $\angle R2 + x + 90° = 180°$; $\angle R2 = 90° - x$; $\angle R1 + \angle RNP + \angle P1 = 180°$, $\angle R1 + 90° + x = 180°$, $\angle R1 = 90° - x$. The conclusion that $\angle R1 = \angle R2 = 90° - x$, PR bisects $\angle OR$, was written from a correct and complete reasoning pattern. For these reasons, Learner F was ranked excellent in adaptive reasoning.

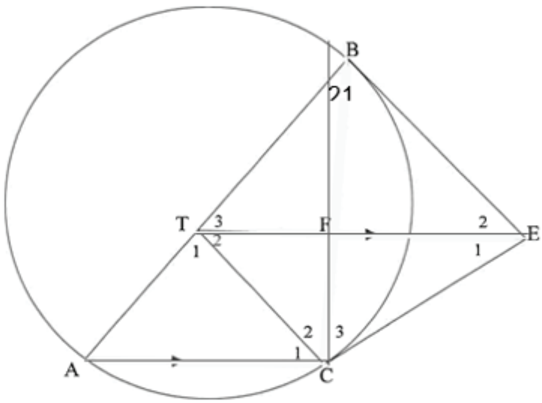
In Question 5.1, learners were asked to prove that $\angle B1 = \angle T3$. The solution to this question should follow the following reasoning pattern:

(1) Find the pattern of the problem with correct reason $\angle C3 = \angle CAB$ (tan-chord);
(2) Present the statement that $\angle B1 = \angle CAB$ (tan-chord);
(3) Deductive reasoning $\angle B1 = \angle C3$ (both $= \angle CAB$);
(4) Establish that $\angle T3 = \angle A = \angle CAB$ (corresponding $\angle$s; TE II AC) and
(5) Correct conclusion and correct reason $\angle T3 = \angle B1$ (both $= \angle CAB$).

Another way to solve this problem was: (1) find the pattern for the problem with correct reason $\angle B1 = \angle A$ (tan-chord); (2) correct statement and correct reason $\angle T3 = \angle A$ (corresponding $\angle$s, TE II AC); (3) deductive reasoning and conclusion: $\angle B1 = \angle T3$ (both $= \angle A$).

Figure 4.3: Question 5.1

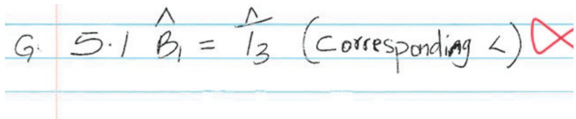
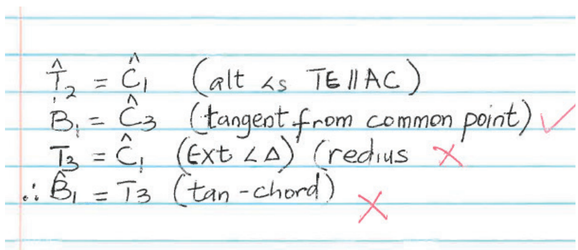
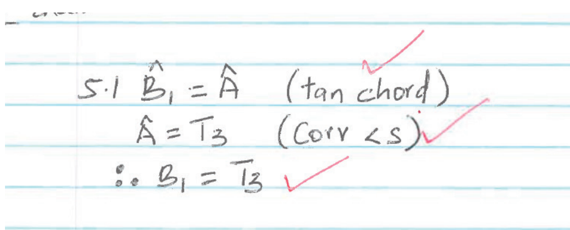
In the diagram, the vertices A, B and C of ΔABC are concyclic. EB and EC are tangents to the circle at B and C respectively. T is a point on AB such that TE II AC. BC cuts TE in F.



5.1. Prove that $\angle B_1 = \angle T_3$.

Table 4.5 shows examples of learners' work at different levels of adaptive reasoning in Question 5.1

Table 4.5: Learner proficiency level and excerpts of solutions to Question 5.1

Learner	**Level of Proficiency**	**Excerpt**
G	**Poor**	
H	**Moderate**	
I	**Excellent**	

The solution of Learner G showed that the learner provided a wrong conjecture, no pattern of the problem and incorrect reasoning by stating $\angle B3 = \angle T3$ with reason that corresponding angles are equal. For these reasons, Learner G was ranked as poor in adaptive reasoning proficiency on Question 5.1. The solution of Learner H showed that the learner's conjecture $\angle B1 = \angle C3$ is less correct, because the reason alternate angles are equal is incorrect. The correct reason is that the 2 angles are subtended by the same chord CE. The proposition that ∠T3 = ∠C1 is also incorrect because ∠T3 = ∠C3 with reason: subtended by the same chord BE; the learner managed to write a conclusion ∠B1= ∠T3, but with the wrong reason of tangent chord theorem; the learner managed to present reasoning even if the pattern of the problem had mistakes. For these reasons, Learner H was ranked moderate in adaptive reasoning in Question 5.1. The solution of Learner I showed that the learner could make the connection between the diagram and the tangent chord theorem to propose that ∠B1 = ∠A, and to establish that ∠T3 = ∠A with reason corresponding angles were equal. Using deductive reasoning to deduce that ∠B1 = ∠T3. For these reasons, Learner I was classified as excellent in adaptive reasoning in Question 5.1.

The findings on the learners' levels of adaptive reasoning from all the questions are presented in Table 4.6. Taking Question 5.1 (row 8) as an example, it was found that 176 learners, accounting for 88 per cent of the participants, were poor in adaptive reasoning: they presented wrong conjectures, no patterns of the problem, incorrect reasoning and without conclusion. Moreover, the learners failed to examine the validity of their arguments. Fourteen learners (7 per cent) were classified as moderate in adaptive reasoning. The learners' conjectures were less complete. The learners managed to write conclusions, however, with wrong reasons; the learners managed to present reasoning, even if the patterns of the problem had mistakes. Ten learners, accounting for 5 per cent of the participants, were excellent in adaptive reasoning: the learners could present correct and complete conjectures; they could find the correct pattern for the solution; they presented a correct and complete reasoning for the solution and were able to examine the validity of their arguments.

Table 4.6: Leaners' adaptive reasoning proficiency from the EG test

Question in EGPT	Frequency					
	Poor		Moderate		Excellent	
	Number[a]	%	Number	%	Number	%
1.2	166	83	10	5	24	12
2.1	158	79	12	6	30	15
2.2	176	88	10	5	14	7
4.1.1	186	93	10	5	4	2
5.1	176	88	14	7	10	5
5.3	194	97	2	1	4	2
5.4	184	92	4	2	12	6
5.5	192	96	2	1	6	3
6.1	160	80	12	6	28	14
6.2	180	90	8	4	12	6
Average percentage		88.6		4.2		7.2

[a]out of 200

The overall performance depicted in Table 4.6 shows that on average, approximately 89 per cent of all learners were poor in adaptive reasoning, while only 4 per cent were found to be moderate, and 7 per cent were qualified as excellent.

Discussion and conclusion

The research question was: What is the level of Grade 11 learners in Vhembe East District's adaptive reasoning in EG problems? This question was addressed by analysing learners' performance in the EGPT from the adaptive reasoning point of view. The findings are that most learners (89 per cent) are poor in adaptive reasoning of EG (Table 4.6). This is specifically in terms of proposing a conjecture, drawing a correct conclusion, finding the patterns of the task, presenting reasoning and examining the validity of an argument. These findings explain the poor performance of learners in EG, a major contribution of the current study.

The findings of the current study resonate with the findings of some existing studies. Susilawati and Dewi (2019) found that more than 50 per cent of high school students in their study exhibited low levels of mathematical reasoning ability. Ali et al. (2014) found that 77 per cent of learners in their study performed poorly in a geometry test. Ngirish and Bansilal (2019) found that 99 per cent of the participants in their study showed no signs of engagement within the formal deduction level, while Ansari et al. (2020) found that most learners in their study lacked adaptive reasoning proficiency.

The current study has practice-related and research-based implications. One practice-related implication is that teachers must assist learners better to increase their adaptive reasoning. This is across all the aspects of adaptive reasoning (Table 4.6). Ansari et al. (2020) concluded that adaptive reasoning proficiency puts learners in a position to think logically and reflectively when addressing mathematical problems. It also helps learners to provide reasons for their solution and helps them link the various patterns before providing a solution. Mabilangan et al. (2011) conclude that problem-solving activities should be embedded in all aspects of learning situations, as exposing the learners to problem-solving tasks can develop learners' mathematical reasoning power and foster their understanding that mathematics is a creative endeavour.

In this regard, however, there may be a need to strengthen teachers' knowledge of teaching problem-solving in South Africa. Several studies have blamed the poor performance of learners in EG on difficulties associated with teaching and learning geometry in South Africa (Alex and Mammen 2018; Chimuka 2017; Dongwi 2014; Patkin and Lavenberg 2012). Some authors have noted that the teaching of problem-solving is weak and unstructured (Ally 2011; Dhlamini and Luneta 2016). Additionally, Bankov (2013) opined that teachers tend to present geometric facts and solutions to learners without providing them opportunities to interact geometrically and develop their reasoning skills. Teachers should attend seminars and workshops on the different dimensions of adaptive reasoning, so that they can better teach these to their learners. The need for preparing lessons that promote all the dimensions of adaptive reasoning has been noted (Ally 2011). Increasing learners' levels of adaptive reasoning will increase their proficiency in solving EG problems. Teachers also need to be able to design formative assessments, targeting the different facets of adaptive reasoning. Teachers should be advised not to present geometric facts and solutions without allowing the learners to interact geometrically and develop their reasoning skills.

The presented study focused on the levels of adaptive reasoning of the participating learners, for the purpose of depth. However, this excluded four other strands of mathematical proficiency—conceptual understanding, procedural fluency, strategic competence and productive disposition. Given the interaction between the different dimensions of mathematical proficiency (Findell et al. 2001), the participating learners' proficiency levels in these other four dimensions may have been impacted by the low level of proficiency in adaptive reasoning. However, their proficiency levels in conceptual understanding, procedural fluency, strategic competence and productive disposition must be studied independently. Additionally, given the focus of the presented study in a single school district, the Vhembe East District in Limpopo province, South Africa, there may be a need to research other school districts in adaptive reasoning and even mathematical proficiency in general, to establish a picture of learner proficiency on a broader scale.

In conclusion, it is worth recalling that the purpose of the presented study was to explore the problem-solving proficiencies of the 200 Grade 11 learners in solving EG problems in terms of adaptive reasoning. The focus was on the Vhembe East District in Limpopo province, South Africa. After evaluating the learners' problem-solving in the EGPT using the RMAR, the findings are that 89 per cent of learners were poor in adaptive reasoning, while only 4 per cent were found to be moderate, and 7 per cent were qualified as excellent in this category. These findings raise questions about the levels of proficiency of the participating learners in other dimensions of mathematical proficiency. The study also raises questions about the mathematical proficiency of learners in other school districts in South Africa. Addressing these questions will allow the teaching and learning of EG to be better supported. Success in mathematics strongly depends on problem-solving proficiency (Ally 2011; Ho 2020), a requirement for learner competitiveness in today's and tomorrow's world (Moodley 2008; Syukriani et al. 2017).

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